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ON THE INFLUENCE OF STRUCTURE OF FORCES ON THE STABILITY OF MOTION

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We investigate the stability of systems as a function of the structure of the forces which may be dissipative, accelerating, gyroscopic, potential and nonconservative [1].

1. Consider the systems

$$x^{\cdot\cdot} + Dx^{\cdot} + Px = 0 \tag{1.1}$$

$$x^{\cdot\cdot} + Dx^{\cdot} + Px = X(x, x^{\cdot}) \tag{1.2}$$

Here and below x is a column matrix with elements $x_1, \ldots, x_n$; $D = D'$, $P = -P' \not\equiv 0$ are constant ($n \times n$)-matrices; $X(x, x^{\cdot})$ is a column-matrix with elements $X_1(x, x^{\cdot}), \ldots, X_n(x, x^{\cdot})$ containing $x_i, x_i^{\cdot}$ in powers not lower than the second, where $X(0, 0) \equiv 0$. The terms $Dx^{\cdot}$ characterize the dissipative and accelerating forces, the terms Px characterize the nonconservative forces, and the terms $X(x, x^{\cdot})$ characterize the nonlinear forces. We follow everywhere the terminology adopted in [1].

About systems (1.1) and (1.2) we know:

1) system (1.1) is not asymptotically stable [2];
2) systems (1.1) and (1.2) are unstable if $D \equiv 0$ [1, 3];
3) systems (1.1) and (1.2) are unstable if $\mathrm{Sp}\, D < 0$ [2];
4) system (1.1) is unstable if D is sign-positive and the determinant $|P| \neq 0$ [3].

In [3] it was asserted that system (1.1) is unstable for an even n and a sign-positive D. However, the proof carried out in [3] is valid only if $|P| \neq 0$ and, moreover, it is valid in this case for an arbitrary constant matrix D.

We consider the characteristic equation (E is the unit matrix)

$$|E\lambda^2 + D\lambda + P| = 0 \tag{1.3}$$

Theorem 1. Characteristic equation (1.3) does not possess roots on the imaginary axis, which are different from zero. If $|P| \neq 0$, Eq. (1.3) has n roots with positive real parts and n roots with negative real parts.

Proof. Let $\lambda = ki$ $(k \neq 0)$ be a root of Eq. (1.3). Then the system

$$(-Ek^2 + Dki + P)\, y = 0$$

has the nontrivial solution $y = a + bi$ and for k we obtain the equation

$$(a - bi)'\,(-Ek^2 + Dki + P)\,(a + bi) = 0$$

or

$$-(a'Ea + b'Eb)\,k^2 + (a'Da + b'Db)\,ki + 2a'Pbi = 0$$

It is impossible to satisfy this equation for any one real $k \neq 0$ since $a'Ea + b'Eb \neq 0$. The first part of the theorem is proved. If $|P| \neq 0$, then the equation

$$|E\lambda^2 + \varepsilon D\lambda + P| = 0 \tag{1.4}$$

has no roots whatsoever on the imaginary axis for any ε. Since the roots of Eq. (1.4) depend continuously on ε while Eq. (1.3) is obtained from the equation

$$|E\lambda^2 + P| = 0 \tag{1.5}$$

by a continuous change of ε in (1.4) from zero to unity, the number of roots of Eq. (1.3) with positive real part coincides with the same for Eq. (1.5) which, as is easily proved, has n roots with a positive real part and n with a negative real part. The theorem is proved.

Corollary 1. Systems (1.1) and (1.2) are unstable if $\mathrm{Sp}D = 0$.

In fact, from Theorem 1 it follows that Eq. (1.3) always has roots with nonzero real part. Since

$$\sum_{i=1}^{2n} \mathrm{Re}\,\lambda_i = -\,\mathrm{Sp}\,D = 0$$

among the roots of characteristic equation (1.3) there always are roots with positive real part, which proves the instability of systems (1.1) and (1.2).

Corollary 2. Systems (1.1) and (1.2) are unstable if $|P| \neq 0$.

Theorem 2. System (1.1) is always unstable.

Proof. If $|P| \neq 0$, the instability of system (1.1) is already proved. We merely note that in this case the instability of system (1.1) can be proved also using the Liapunov's first instability theorem by considering the function

$$V = x'Ex^{\boldsymbol\cdot} + \frac{1}{2}x'Dx + \varepsilon x'Px^{\boldsymbol\cdot} + \frac{\varepsilon}{2}x^{\boldsymbol\cdot\prime}Dx^{\boldsymbol\cdot}$$

whose total time derivative by virtue of system (1.1)

$$V^{\boldsymbol\cdot} = x^{\boldsymbol\cdot\prime}\,(E - \varepsilon D^2)\,x^{\boldsymbol\cdot} + \varepsilon\,(Px)'\,Px$$

is positive-definite for a sufficiently small $\varepsilon > 0$.

Suppose that $|P| = 0$. In this case, system (1.1) can be written in the form [4]

$$y^{\boldsymbol\cdot\boldsymbol\cdot} + D_1 y^{\boldsymbol\cdot} + P_1 y = 0 \tag{1.6}$$

$$P_1 = \left\| \begin{matrix} P_{rr} & P_{rq} \\ P_{qr} & P_{qq} \end{matrix} \right\| = -P_1'$$

by means of the transformation $x = Ty$, where T is a constant orthogonal matrix. Here the blocks P_{rr}, P_{rq}, P_{qr} are identically zero, while $|P_{qq}| > 0$; the subscripts r, q indicate the dimensions of the blocks; $r + q = n$. Let

$$D_1 = \left\| \begin{matrix} D_{rr} & D_{rq} \\ D_{qr} & D_{qq} \end{matrix} \right\|$$

in correspondence with P_1 :

a) $|D_{rr}| \neq 0$. In this case it can be shown that Eq. (1.4) has precisely r zero roots for any $\varepsilon \neq 0$. Equation (1.5) has q roots with positive real part and, therefore, characteristic equation (1.3) also has at least q roots with positive real part, which can be perceived by changing ε from zero to unity in Eq. (1.4). Thus, system (1.1) is unstable in this case too.

b) $|D_{rr}| = 0$. In this case system (1.6) has solutions of the form

$$y = at + b \tag{1.7}$$

where the constant column-matrices a and b are chosen as follows. The column-matrix a has the form $a = (a_r, 0)$, where a_r is a nonzero solution of the system $D_{rr}z = 0$. The column-matrix b is a solution of the system $P_1 z + D_1 a = 0$ which in the case under consideration always has a nonzero solution. The presence of solutions of form (1.7) proves the instability of system (1.6), and, hence, of system (1.1) when $|P| = 0$, $|D_{rr}| = 0$. The theorem is proved.

Corollary. System (1.2) is unstable when $|P| = 0$, $|D_{rr}| \neq 0$.

Remaining uninvestigated is the possibility of stabilizing system (1.1) by nonlinear forces $X(x, x^{\cdot})$ in the case when $|P| = 0$, $|D_{rr}| = 0$, $\mathrm{Sp}\, D > 0$.

Example 1. Let matrix D in system (1.2) be positive definite. Then system (1.2) is unstable. In fact, in this case $|D_{rr}| > 0$ for any possible r, being the principal diagonal minor of a positive-definite matrix D_1, and, in accordance to what we have proved, system (1.2) is unstable.

2. Consider the systems

$$x^{\cdot\cdot} + Gx^{\cdot} + Px = 0 \tag{2.1}$$

$$x^{\cdot\cdot} + Gx^{\cdot} + Px = X(x, x^{\cdot}) \tag{2.2}$$

where G is a constant skew-symmetric ($n \times n$)-matrix and the matrix P is the same as in system (1.1). The terms $Gx^{\cdot}$ characterize gyroscopic forces. About systems (2.1) and (2.2) we know:

1) system (2.1) is not asymptotically stable [2];

2) system (2.1) with $P \equiv 0$ is stable if and only if $|G| \neq 0$ [2].

Consider the characteristic equation

$$|E\lambda^2 + G\lambda + P| = 0 \tag{2.3}$$

Theorem 3. Systems (2.1) and (2.2) are unstable if $G = kP$, $P \not\equiv 0$, k is an arbitrary constant.

Proof. As in the proof of Theorem 1 we can show that in the case given Eq. (2.3) does not possess roots on the imaginary axis, which are different from zero. This together with the condition $\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 + \ldots + \operatorname{Re}\lambda_{2n} = 0$ proves the theorem.

Corollary. Systems (2.1) and (2.2) with $n = 2$ and $P \not\equiv 0$ are always unstable.

Theorem 4. System (2.1) is unstable if matrices G and P commute and $P \not\equiv 0$.

This theorem is a corollary of Chetaev's instability theorem [5] which is satisfied by the function

$$V = x'Px^{\cdot} + \frac{1}{2}x'PGx$$

Theorem 5. System (2.2) is unstable if the matrices G and P commute and $|P| \neq 0$.

Proof. Consider the function

$$V = \varepsilon x' Ex^{\cdot} + x'Px^{\cdot} + \frac{1}{2}x'PGx \tag{2.4}$$

whose total time derivative by virtue of system (2.2) is

$$V^{\cdot} = \varepsilon\left(x^{\cdot} + \frac{1}{2}Gx\right)'\left(x^{\cdot} + \frac{1}{2}Gx\right) + \frac{\varepsilon}{4}x'GGx + (Px)'Px + x'(\varepsilon E + P)X(x, x^{\cdot})$$

Obviously, function (2.4) satisfies the hypotheses of Liapunov's insability theorem for a sufficiently small $\varepsilon > 0$; this proves the theorem.

Theorem 6. System (2.1) is always unstable for odd n.

Proof. The determinants $|G| = |P| = 0$ for odd n. As in the proof of Theorem 2 we can show that in this case system (2.1) always has solutions of form (1.7), and that proves the theorem.

We consider three examples.

Example 2. The system

$$\begin{aligned} &x_1^{\cdot\cdot} + g_1x_2^{\cdot} + g_2x_3^{\cdot} = 0 \\ &x_2^{\cdot\cdot} - g_1x_1^{\cdot} + g_3x_3^{\cdot} + px_3 = 0 \\ &x_3^{\cdot\cdot} - g_2x_1^{\cdot} - g_3x_2^{\cdot} - px_2 = 0 \end{aligned} \tag{2.5}$$

where $p \neq 0$, covers [4] all systems of form (2.1) when $n = 3$. According to Theorem 6 it is unstable since it has solutions of form (1.7)

$$x_1 = at + b, \quad x_2 = -ag_2/p, \quad x_3 = ag_1/p$$

where $a \neq 0$. The characteristic equation of system (2.5) with $g_3 \neq 0$ has roots with positive real part; this implies the impossibility of stabilizing it by nonlinear forces. However, if $g_3 = 0$, such a stabilization is possible under the condition $g_1^2 + g_2^2 > 4p^2$.

Example 3. Consider system (2.1) in which

$$G = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}, \qquad P = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{Vmatrix}$$

If we set $P \equiv 0$ or $G \equiv 0$, we obtain, respectively, a purely gyroscopic unstable ($|G| = 0$) system or a purely nonconservative unstable system. The system's characteristic equation is

$$\lambda^2 (\lambda^2 + 2)(\lambda^2 + 3)(\lambda^2 + 6) = 0$$

Obviously, instability can occur only in connection with a double zero root, i.e. because of the possibility of the existence of a solution of form (1.7)

$$x_i = a_i t + b_i \quad (i = 1, \ldots, 4), \qquad \sum_{i=1}^{4} a_i^2 \neq 0$$

Substituting this solution into the system, we find that $a_1 = a_2 = a_3 = a_4 = 0$. Thus the system is stable. Example 3 proves the possibility of a gyroscopic stabilization of a purely nonconservative unstable system in the case $|P| = 0$ and the possibility of stabilizing a purely gyroscopic unstable system by nonconservative forces.

Example 4. Consider system (2.1) in which

$$G = \begin{Vmatrix} 0 & 0 & 0 & g^+ \\ 0 & 0 & g^- & 0 \\ 0 & -g^- & 0 & 0 \\ -g^+ & 0 & 0 & 0 \end{Vmatrix}, \qquad P = \begin{Vmatrix} 0 & p_1 & 0 & 0 \\ -p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_2 \\ 0 & 0 & -p_2 & 0 \end{Vmatrix}$$

Here we set

$$g^{\pm} = \sqrt{11 \pm \frac{\sqrt{20010}}{24}}, \qquad p_1 = \frac{\sqrt{1991 + \sqrt{425137}}}{8\sqrt{3}}, \qquad p_2 = 4\sqrt{6}\, p_1^{-1}$$

Then the system's characteristic equation

$$(\lambda^2 + 1)(\lambda^2 + 2)(\lambda^2 + 3)(\lambda^2 + 16) = 0$$

has distinct purely-imaginary roots, which implies that the system is stable. This example proves the possibility of gyroscopic stabilization of a purely nonconservative unstable system when $|P| \neq 0$.

Examples 3 and 4 refute the assertion in [6] that it is impossible to stabilize a purely nonconservative system by gyroscopic forces alone.

3. Consider the systems

$$x^{\cdot\cdot} + Dx^{\cdot} + Fx + Px = 0 \tag{3.1}$$

$$x^{\cdot\cdot} + Dx^{\cdot} + Fx + Px = X(x, x^{\cdot}) \tag{3.2}$$

where $F = F' \not\equiv 0$ is a constant ($n \times n$)-matrix, while matrices D and P are the same as in system (1.1).

About systems (3.1) and (3.2) we know:

1) systems (3.1) and (3.2) are unstable if $\operatorname{Sp} D < 0$ [2];

2) systems (3.1) and (3.2) are unstable for odd n if F is negative definite [2];

3) systems (3.1) and (3.2) are unstable if n is even, F is negative definite, and D is positive definite [2];

4) statement (3) is valid also for a sign-positive D [3];

5) if $D \equiv 0$, system (3.1) is not asymptotically stable but can be stable [1, 2];

6) systems (3.1) and (3.2) are unstable if $D \equiv 0$ and the Poincaré coefficients are equal to each other [1, 3].

The proof of statement (3) suggested in [2] is valid also for a sign-positive D ($D \not\equiv 0$). The proof of statement (4) suggested in [3] is valid for any constant matrix D.

Consider the characteristic equation

$$| E\lambda^2 + D\lambda + F + P | = 0 \tag{3.3}$$

Theorem 7. If matrix F is negative definite, then Eq. (3.3) has half its roots with positive real part and half with negative part.

Proof. As in the proof of Theorem 1 we show that the equation

$$| E\lambda^2 + \varepsilon D\lambda + F + \varepsilon P | = 0 \tag{3.4}$$

does not have roots on the imaginary axis for any real ε if F is negative definite. Varying ε in Eq. (3.4) from zero to unity, we establish the theorem's validity.

Corollary. Systems (3.1) and (3.2) are unstable if F is negative definite.

Suppose that the matrices F and P in system (3.2) depend on x, $x^{\cdot}$, t.

Theorem 8. If the matrix $F\ (x,\ x^{\cdot},\ t)$ satisfies the generalized Sylvester criterion [1] for a negative-definite quadratic form, then system (3.2) is unstable.

Proof. The validity of this theorem can be established using Liapunov's first instability theorem which is satisfied by the function

$$V = x'Ex^{\cdot} + \frac{1}{2}x'Dx$$

Example 5. Consider the system

$$\begin{aligned} &x_1^{\cdot\cdot} + b_1x_1^{\cdot} + c_1x_1 = 0 \\ &x_2^{\cdot\cdot} + b_2x_2^{\cdot} + c_2x_2 = 0 \\ &b_1b_2 < 0,\ c_1 < 0,\ c_2 < 0 \end{aligned} \tag{3.5}$$

The possibility of a gyroscopic stabilization of the unstable system (3.5) was shown in [2] on the example of the gyro-stabilized monorail carriage. If follows from the corollary of Theorem 7 that system (3.5), as well as a monorail carriage, cannot be stabilized only by nonconservative forces, when the gyroscope is at rest (independently of the nonlinear terms $X\ (x,\ x^{\cdot})$).

Theorem 9. System (3.1) is unstable if $D \equiv 0$ and $\mathrm{Sp}\ F \leqslant 0$.

Proof. Let us examine the system's characteristic equation which in the case $D \equiv 0$ has the form

$$\lambda^{2n} + a_2\lambda^{2n-2} + \ldots + a_{2n} = 0 \tag{3.6}$$

We can show that $a_2 = \mathrm{Sp}\ F$. If $\mathrm{Sp}\ F < 0$, Eq. (3.6) must have roots with positive real part, which proves the theorem in this case. Let $\mathrm{Sp}\ F = 0$ and let system (3.1) be stable. Then all roots of Eq. (3.6) must have the form $\lambda_j = \pm k_ji\ (j = 1, \ldots, n)$, while the equation itself can be written as

$$(\lambda^2 + k_1^2)(\lambda^2 + k_2^2) \ldots (\lambda^2 + k_n^2) = 0$$

whence we obtain

$$k_1^2 + k_2^2 + \ldots + k_n^2 = a_2 = \mathrm{Sp}\ F = 0$$

From the last equality it follows that under the assumptions made all the roots of Eq. (3.6) equal zero, but then it is easily shown that system (3.1) is unstable. The contradiction obtained also proves the theorem in this case.

Corollary. System (3.2) is unstable if $D \equiv 0$ and $\mathrm{Sp}\ F < 0$.

Problem. Under what conditions can the unstable system

$$x^{\cdot\cdot} + Fx = 0 \tag{3.7}$$

be stabilized by nonconservative forces, i.e. how can we select a constant skew-symmetric matrix P so that the system

$$x^{\cdot\cdot} + Fx + Px = 0 \tag{3.8}$$

is stable?

From Theorem 9 it follows that such a stabilization can be possible only under the condition

$$\mathrm{Sp}\, F > 0 \tag{3.9}$$

We see from the results presented in [1] (pp. 198, 199) that the following theorem holds.

Theorem 10. The unstable system (3.7) with $n = 2$ can be stabilized by nonconservative forces if and only if inequality (3.9) is fulfilled.

We have proved below that Theorem 10 is also valid for $n = 3$. In fact, without loss of generality, in systems (3.7) and (3.8) we set

$$F = \begin{Vmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{Vmatrix}, \qquad P = \begin{Vmatrix} 0 & p_1 & p_2 \\ -p_1 & 0 & p_3 \\ -p_2 & -p_3 & 0 \end{Vmatrix}$$

where the inequalities

$$f_1 \geqslant f_2 \geqslant f_3, \qquad f_3 \leqslant 0 \tag{3.10}$$

hold. If the inequality $f_2 + f_3 > 0$ is fulfilled, then we set $p_1 = p_2 = 0$ and the question reduces to the already considered case of $n = 2$. Therefore, it is sufficient to examine the case

$$f_2 + f_3 \leqslant 0 \tag{3.11}$$

The characteristic equation of system (3.8) is

$$f(y) = y^3 + a_1 y^2 + a_2 y + a_3 = 0 \quad (y = \lambda^2) \tag{3.12}$$

where

$$a_1 = f_1 + f_2 + f_3, \quad a_2 = p_1^2 + p_2^2 + p_3^2 + f_1 f_2 + f_1 f_3 + f_2 f_3 \tag{3.13}$$
$$a_3 = f_3 p_1^2 + f_2 p_2^2 + f_1 p_3^2 + f_1 f_2 f_3$$

The system is stable if Eq. (3.12) has distinct negative roots. We set

$$a_2 = {}^1/_4 a_1^2, \qquad a_3 = {}^1/_{108} a_1^3 \tag{3.14}$$

Then we can verify that all the conditions for the negativity of the roots of Eq. (3.12) [7] are fulfilled if inequality (3.9) is fulfilled. Direct verification shows that the roots of the equation $f'(y) = 0$ are not roots of Eq. (3.12), i.e. all roots of Eq. (3.12) are distinct. Thus, system (3.8) is stable if conditions (3.9), (3.14) are fulfilled. Setting $p_2 = 0$ and using (3.13), condition (3.14) can be written as a linear system in p_1^2, p_3^2. When conditions (3.9), (3.11) are fulfilled this system has a unique positive solution, which completes the proof of Theorem 10 for $n = 3$.

We note that the question of stabilizing a purely nonconservative system (3.8) with $F \equiv 0$ by potential forces reduces [4] to the case of $n = 2$ and always has a positive solution, i.e. the following theorem is valid.

Theorem 11. A purely nonconservative system can always be stabilized by potential forces.

Example 6. A particle of unit mass is elastically connected with the axes of a fixed coordinate system $Oxyz$. The particle's equations of motion are

$$x^{\cdot\cdot} + f_1 x = 0,\ y^{\cdot\cdot} + f_2 y = 0,\ z^{\cdot\cdot} + f_3 z = 0$$

The origin O is a trivial equilibrium position, unstable under conditions (3.10), (3.11). Let us assume that condition (3.9) is fulfilled. Then, according to what we have proved above, this equilibrium position can be made stable if to the potential forces acting on the particle we add a nonconservative force P perpendicular to the particle's radius-vector with projections on the coordinate axes $P_x = -p_1 y$, $P_y = p_1 x - p_3 z$, $P_z = p_3 y$, where p_1, p_3 satisfy system (3.14) $(p_2 = 0)$.

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THE STRUCTURE OF SHOCK WAVES IN HYPERSONIC FLOWS

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The motion of gas in that region of curved hypersonic shock wave, where the angle of inclination τ of the latter to the velocity vector of the unperturbed stream is small, is analyzed with the use of Navier-Stokes equations. The number of terms retained in expansions of unknown functions in powers of τ is such as to permit the extension of solution into a new inviscid region by using the method of matching outer and inner asymptotic expansions. The statement of the problem in the new region is distinguished by that functions are specified at a point not by their values but by Taylor series.

1. Basic estimates and the form of asymptotic expansions in the region of the shock wave for $x \to \infty$. Let us consider the hypersonic flow of perfect gas of constant specific heats c_p and c_v. We denote the density of gas